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Uniqueness Results for a Class of Nonlinear Problems*

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Abstract. The purpose of this paper is to give a simple uniqueness result for a class of nonlinear partial differential equations. As a particular case, we obtain the uniqueness of the solution for the problem studied in [1], where this question had been left open.

1. ORIGINAL MOTIVATIONS

Let us briefly recall the problem studied in [1]. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain, with smooth boundary $\Gamma = \partial\Omega$. Given $f \in L^2(\Omega), \mathbf{b} \in [W^{1,\infty}(\Omega)]^n$, $c \in L^\infty(\Omega)$ and a constant viscosity $\nu > 0$, consider the following convection–diffusion problem:

$$\begin{cases} -\nu\Delta u + \mathbf{b}\cdot\nabla u + cu = f \text{ in } \Omega \\ \qquad\qquad\qquad\qquad u = 0 \text{ on } \Gamma. \end{cases} \tag{1.1}$$

If we assume, e.g., that $c(x) \geq 0$, then it is well–known that problem (1.1) has a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$.

Consider now the function $\alpha : \mathbf{R} \to \mathbf{R}$ defined by

$$\alpha(s) = \begin{cases} 0 & 0 \leq s \leq \delta - \sigma \\ \dfrac{\delta}{\sigma}(s-\delta) + \delta & \delta - \sigma < s < \delta \\ s & s \geq \delta \end{cases}$$

$$\alpha(s) = -\alpha(-s) \qquad\qquad s < 0 \tag{1.2}$$

where δ and σ are two positive numbers and $\sigma < \delta$. Clearly α is a monotonically increasing continuous function (see Fig. 1.1).

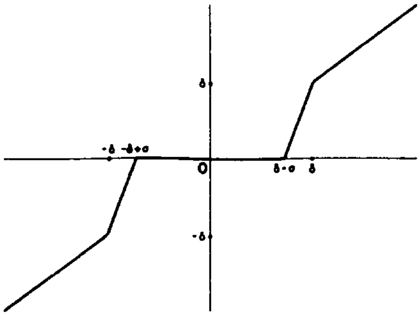

Fig. 1.1

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In [1] we studied the problem

$$\begin{cases} -\nu\alpha(\Delta u) + \mathbf{b}\cdot\nabla u + cu = f & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \end{cases} \tag{1.3}$$

as an approximation of problem (1.1), proving existence of a solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and an error estimate for the difference between the solution of (1.3) and the solution of (1.1). The latter result holds under the following crucial hypothesis on $\mathbf{b}(x)$:

HYPOTHESIS 1.1. *There exist a constant vector $\mathbf{k} \in \mathbf{R}^n$ and a constant number $C > 0$ such that*

$$\mathbf{b}(x)\cdot\mathbf{k} \geq C > 0 \qquad \forall x \in \Omega.$$

We will now see that Hypothesis 1.1 together with some regularity assumptions implies the uniqueness of the solution for problem (1.3).

2. STATEMENT OF THE RESULT

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with a smooth boundary $\Gamma = \partial\Omega$. Let A be the second-order linear elliptic operator given by

$$Au = \sum_{i,j} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \mathbf{a}\cdot\nabla u$$

where $a_{ij} \in C^0(\overline{\Omega}), \mathbf{a} \in [C^0(\overline{\Omega})]^n$ and $a_{ij}(x)$ satisfy the usual condition of uniform ellipticity. With these hypotheses, the operator A satisfies the weak maximum principle, (see for instance [2], Chapter 8). Let B be the first-order linear differential operator given by $Bu = \mathbf{b}\cdot\nabla u + cu$ where $\mathbf{b} \in [C^0(\overline{\Omega})]^n$, $c \in C^0(\overline{\Omega})$ and $c(x) \geq 0$ for any $x \in \Omega$. We assume that Hypothesis 1.1 holds for $\mathbf{b}$, that is, there exists a constant vector $\mathbf{k}$ and a constant number $C > 0$ such that $\mathbf{b}(x)\cdot\mathbf{k} \geq C > 0 \quad \forall x \in \Omega$. In addition, we assume that

$$\mathbf{a}(x)\cdot\mathbf{k} \leq 0 \quad \forall x \in \Omega. \tag{2.1}$$

Let $\alpha : \mathbf{R} \to \mathbf{R}$ be a monotone increasing function and f, g functions defined respectively in Ω and on Γ. We then have the following theorem:

THEOREM 2.1. *The problem*

$$\begin{cases} -\alpha(Au) + Bu = f & \text{in } \Omega \\ u = g & \text{on } \Gamma \end{cases} \tag{2.2}$$

has at most one solution $u \in C^1(\Omega) \cap C^0(\overline{\Omega}) \cap H^2(\Omega)$.

PROOF: Let u_1 and u_2 be two solutions of problem (2.2), and let $w = u_1 - u_2$. We have

$$\begin{cases} -\alpha(Au_1) + Bu_1 = f & \text{in } \Omega \\ u_1 = g & \text{on } \Gamma \end{cases} \tag{2.3}$$

and

$$\begin{cases} -\alpha(Au_2) + Bu_2 = f & \text{in } \Omega \\ u_2 = g & \text{on } \Gamma. \end{cases} \tag{2.4}$$

By subtracting (2.4) from (2.3) and multiplying by $Au_1 - Au_2 = Aw$ we get the equation

$$\begin{cases} -(\alpha(Au_1) - \alpha(Au_2))(Au_1 - Au_2) + Aw\,Bw = 0 & \text{in } \Omega \\ w = 0 & \text{on } \Gamma. \end{cases}$$

Now using the monotonicity of α we have

$$-(\alpha(Au_1) - \alpha(Au_2))(Au_1 - Au_2) \leq 0$$

so we get the inequality

$$\begin{cases} Aw\,Bw \geq 0 & \text{in } \Omega \\ w = 0 & \text{on } \Gamma. \end{cases} \tag{2.5}$$

We will actually prove that (2.5) implies $w(x) = 0$ for any $x \in \Omega$.

It is clearly enough to show that (2.5) implies $w \leq 0$. In fact, if (2.5) holds for w, it holds also for $-w$, so from the previous argument we obtain $-w \leq 0$, that is $w \geq 0$; then we have $w \equiv 0$.

In order to show that (2.5) implies $w \leq 0$ in Ω, we reason by contradiction. Suppose that

$$\max_{x \in \Omega \cup \Gamma} w(x) = w(x_{\max}) = M > 0.$$

The idea is to perturbate w with an affine function and a quadratic one in order to obtain a contradiction with the maximum principle.

To this end, let w_1 be an affine function such that

$$\begin{cases} w_1(x_{\max}) = 0 & \\ \mathbf{b}(x) \cdot \nabla w_1(x) \geq C_1 > 0 & \forall x \in \Omega \\ \mathbf{a}(x) \cdot \nabla w_1(x) \leq 0 & \forall x \in \Omega \\ \|w_1\|_{L^\infty(\Omega)} \leq M/3 & \end{cases} \tag{2.6}$$

for some positive constant C_1. The function w_1 can be constructed starting from $\epsilon_1 \mathbf{k} \cdot (x - x_{\max})$ and then tuning ϵ_1 ($\mathbf{k}$ has the property stated in Hypothesis 1.1 and (2.1) holds). Then let w_2 be a smooth function such that

$$\begin{cases} w_2(x_{\max}) = 0 \\ \|w_2\|_{L^\infty(\Omega)} \leq \frac{M}{3} \\ \|\nabla w_2\|_{L^\infty(\Omega)} \leq \frac{C_1}{2\|\mathbf{b}\|_{L^\infty(\Omega)}} \\ Aw_2(x) \geq C_2 > 0 \end{cases} \tag{2.7}$$

for some positive constant C_2. The function w_2 can be constructed for instance starting from a smooth function v such that $Av \geq C_3 > 0$ for some positive constant C_3, then defining $w_2 = \epsilon_2 v$ with a sufficiently small ϵ_2.

We define finally $\overline{w}$ by

$$\overline{w} = w - w_1 + w_2.$$

We show now that the maximum point for $\overline{w}$ is in the interior of Ω. Let $y \in \Gamma$; then

$$\overline{w}(y) = w(y) - w_1(y) + w_2(y) = -w_1(y) + w_2(y) < \frac{2}{3}M$$

so $\overline{w}(y) < M = \overline{w}(x_{\max})$. Let now $\overline{x}_{\max}$ be a maximum point for $\overline{w}$:

$$\overline{w}(\overline{x}_{\max}) = \max_{x \in \Omega \cup \Gamma} \overline{w}(x).$$

We have

$$w(\overline{x}_{\max}) = \overline{w}(\overline{x}_{\max}) + w_1(\overline{x}_{\max}) - w_2(\overline{x}_{\max}) \geq \frac{M}{3} \tag{2.8}$$

since $\overline{w}(\overline{x}_{\max}) \geq \overline{w}(x_{\max}) = M$ and $|w_1(\overline{x}_{\max})| < M/3$, $|w_2(\overline{x}_{\max})| < M/3$ by (2.6) and (2.7).

Since $\overline{x}_{\max}$ is an interior maximum point for $\overline{w}$ and $\overline{w}$ is a C^1 function, we have

$$\nabla\overline{w}(\overline{x}_{\max}) = 0$$

and so

$$\nabla w(\overline{x}_{\max}) = \nabla w_1(\overline{x}_{\max}) - \nabla w_2(\overline{x}_{\max}).$$

Multiplying by $\mathbf{b}(\overline{x}_{\max})$ and adding $c(\overline{x}_{\max})w(\overline{x}_{\max})$ we obtain

$$\begin{aligned} Bw(\overline{x}_{\max}) &= \mathbf{b}(\overline{x}_{\max})\cdot\nabla w(\overline{x}_{\max}) + c(\overline{x}_{\max})w(\overline{x}_{\max}) = \\ &= \mathbf{b}(\overline{x}_{\max})\cdot\nabla w_1(\overline{x}_{\max}) + c(\overline{x}_{\max})w(\overline{x}_{\max}) - \mathbf{b}(\overline{x}_{\max})\cdot\nabla w_2(\overline{x}_{\max}) \end{aligned}$$

and using (2.6), (2.7), the positiveness of c and (2.8) we end up with

$$Bw(\overline{x}_{\max}) \geq \frac{C_1}{2} > 0.$$

By continuity, there exists a neighborhood $U_{\overline{x}_{\max}} \subset \Omega$ of $\overline{x}_{\max}$ such that

$$Bw(x) > 0 \qquad \forall x \in U_{\overline{x}_{\max}}.$$

Now we can use our main hypothesis in (2.5) and conclude that

$$Aw(x) \geq 0 \qquad \forall x \in U_{\overline{x}_{\max}}.$$

Recalling the definition of $\overline{w}$, we have

$$A\overline{w} = Aw - Aw_1 + Aw_2 = Aw - \mathbf{a}\cdot\nabla w_1 + Aw_2 \geq Aw_2$$

so by (2.7)

$$\begin{cases} A\overline{w}(x) \geq C_2 > 0 & \forall x \in U_{\overline{x}_{\max}} \\ \overline{x}_{\max} \text{ is a maximum point for } \overline{w}. \end{cases}$$

Since by hypothesis the operator A satisfies a weak maximum principle, we have a contradiction. So, by (2.5), $w(x) \leq 0$ for any $x \in \Omega$ and the theorem is proved. ∎

It is very easy to find operators A and B such that Hypothesis 1.1 is not satisfied and condition (2.5) does not imply $w \equiv 0$. For example, let $n = 1$, $\Omega =]-\frac{\pi}{2}, \frac{\pi}{2}[$, $Au = u''$, $Bu = xu'$. Clearly $u(x) = \cos x$ satisfies (2.5) since $xu'u'' = \frac{x}{2}\sin 2x \geq 0$ in $]-\frac{\pi}{2}, \frac{\pi}{2}[$, but u is not identically zero on $]-\frac{\pi}{2}, \frac{\pi}{2}[$.

3. CONCLUSIONS

Going back to our original problem (1.3), it is interesting to note that Hypothesis 1.1 implies both the fact that the solution of (1.3) is unique and that it is a good approximation of the solution of (1.1). This non-degeneracy of the vector field $\mathbf{b}(x)$ is necessary in order to have a convection-dominated problem where diffusion plays an important role only in a small region of the domain (the boundary and internal layers) which shrinks when the viscosity ν tends to zero.

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